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SUMMARY OF "THEORIE ANALYTIQUE DES PROBLEMES STOCHASTIQUES RELATIFS A UN GROUPE DE LIGNES TELEPHONIQUES AVEC DISPOSITIF D'ATTENTE" *

by

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Introduction

The purpose of this memoir is to treat in the utmost generality the most obvious queueing problems concerning a bundle of s fully available telephone trunks (or a system of s equally accessible counters) where calls (or customers) are treated according to the rule "first come, first served".

Although being concerned with probability problems, we don't borrow from probability calculus anything but the very notions of probability, of dependent or independent chance variables and of distribution functions (d. f.) of one or several variables. As our methods are exclusively analytical ones, we employ widely the elementary parts of the theory of analytical functions of one complex variable.

In general we make the following assumptions:

For all holding times T_n and interarrival intervals (of successive calls) Y_n $(n=0,1,2,\ldots)$ the probabilities

$$\mathrm{Prob}\,(T_m < t) = f_1(t) \qquad \mathrm{Prob}\,(Y_m < y) = f_2(y)$$

are arbitrarily given distribution functions, the only restrictive hypothesis being the condition

$$E(T) = \int_0^\infty t \, df_1(t) < \infty .$$

As initial condition we suppose that from the instant of generation of a given call (which is denoted as call "0") on, there elapse respectively $t_{01} \gtreqless 0, \cdots, t_{0s} \gtreqless 0$ units of time, before all s trunks are free from previously generated calls.

Then one of our basic results runs as follows. Let τ_n be the waiting time of the n th call, q $(R(q) > 0)$ a complex parameter and γ $(|\gamma| < 1)$ a complex variable. The generating function of the mathematical expectations $E e^{-q\tau_n}$ i. e.

$$\Phi(q,\gamma) = \sum_{n=0}^{\infty} \gamma^n E e^{-q\tau_n}$$

is given by the following expression:

$$(1) \quad \Phi(q,\gamma) = V_0(0) + \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \sum_{\nu=1}^{s} e^{z_1 t_{0\nu}} \cdot V_1(z_1; z_1) \frac{dz_1}{z_1} +$$

$$+ \frac{1}{(2\pi i)^2} \int_{-i\infty+\delta}^{i\infty+\delta} \int_{-i\infty+\delta}^{i\infty+\delta} \sum_{\nu_1<\nu_2} e^{z_1 t_{0\nu_1} + z_2 t_{0\nu_2}} \cdot V_2(z_1, z_2; z_1+z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

$$+ \cdots + \frac{1}{(2\pi i)^s} \int_{-i\infty+\delta}^{i\infty+\delta} \cdots \int_{-i\infty+\delta}^{i\infty+\delta} e^{\sum_{\nu=1}^{s} z_\nu t_{0\nu}} \cdot V_s\left(z_1, \cdots, z_s; \sum_{\nu=1}^{s} z_\nu\right) \frac{dz_1 \cdots dz_s}{z_1 \cdots z_s}$$

$$(\delta > 0)$$

where the functions V_i are given by the system of s linear simultaneous integral equations

$$(2)\quad \left[1-\gamma\,\varepsilon_2(-y)\right]V_\lambda(\gamma_1,\cdots\gamma_\lambda;y)-\frac{\gamma}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\varepsilon_1(\zeta)-1}{\zeta}\cdot\frac{1}{y-\sum_1^\lambda\gamma_\nu-\zeta}\left[\left(y-\sum_1^\lambda\gamma_\nu\right)\varepsilon_2(-y)\cdot\right.$$

$$\left.\cdot V_{\lambda+1}(\gamma_1,\cdots,\gamma_\lambda,\zeta;y)-\zeta\,\varepsilon_2\left(-\sum_1^\lambda\gamma_\nu-\zeta\right)V_{\lambda+1}\left(\gamma_1,\cdots,\gamma_\lambda,\zeta;\sum_1^\lambda\gamma_\nu+\zeta\right)\right]d\zeta$$

$$=\delta_{\lambda 0}\frac{q}{q+y}\qquad\qquad(\lambda=0,1,\cdots,s-1)$$

and the equation

$$(3)\qquad \sum_{\lambda=0}^{s}\ \sum_{1',\cdots,\lambda'=1}^{s}V_\lambda(\gamma_{1'},\cdots,\gamma_{\lambda'};y)=0\ .$$

Here $\sum_{1',\cdots,\lambda'=1}^{s}$ has to be extended over all $\binom{s}{\lambda}$ combinations of the s indices $1,\cdots,s.$ $\varepsilon_1(\gamma)$ and $\varepsilon_2(\gamma)$ are respectively defined by

$$\varepsilon_i(\gamma)=\int_0^\infty e^{\gamma t}\,df_i(t)\qquad(i=1,2\quad R(\gamma)\leq 0)$$

while

$$\delta_{\lambda 0}=\begin{cases}1 & \lambda=0\\ 0 & \lambda\neq 0\end{cases}\ .$$

The functions $V_\lambda(\gamma_1, \cdots, \gamma_\lambda; \gamma)$ are analytic for

$$R(\gamma_1) > 0$$
$$\vdots$$
$$R(\gamma_\lambda) > 0$$
$$R(\gamma) > 0$$

and symmetric in $\gamma_1, \cdots, \gamma_\lambda$. In all integral formulae similar to (1) these functions appear only in the form

$$V_\lambda(\gamma_1, \cdots, \gamma_\lambda; \sum_{\nu=1}^{\lambda} \gamma_\nu).$$

The system of integral equations (2), (3) can be solved by a finite number of steps in the following two cases:

1.° For $\mathcal{E}_1(\gamma)$ rational, i. e., when the d. f. $f_1(t)$ is of the form

$$f_1(t) = 1 - \sum P_\nu(t) e^{-\alpha_\nu t} \qquad (R(\alpha_\nu) > 0),$$

where $P_\nu(t)$ is a polynomial in t, $f_2(t)$ (and therefore $\mathcal{E}_2(\gamma)$) being an arbitrary function.

2.0 For $\varepsilon_1(\gamma)$ of the form $P_m(e^{h\gamma})$, i. e. when $f_1(t)$ is a lattice function with distance h, $\varepsilon_2(\gamma)$ being rational.

As is known by the theorem of Paul Lévy, which in the present case, owing to the fact that $\tau_m \geqq 0$, can be written in a simpler form, the d.f. $\rho_m(t) = \mathrm{Prob}(\tau_m < t)$ can be derived from $E\, e^{-q\tau_m}$ by the formula

$$(4) \qquad \rho_m(t) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{-iN+\delta}^{iN+\delta} e^{qt}\, E\, e^{-q\tau_m} \frac{dq}{q} \qquad (t>0,\ \delta>0).$$

More generally generating functions like

$$\sum_{m,m'=0}^{\infty} \gamma^m \gamma'^{m'} E\, e^{-q\tau_m - q'\tau_{m+m'}},$$

$$\sum_{m,m',m''=0}^{\infty} \gamma^m \gamma'^{m'} \gamma''^{m''} E\, e^{-q\tau_m - q'\tau_{m+m'} - q''\tau_{m+m'+m''}}, \cdots$$

$$(R(q)>0,\ R(q')>0,\ R(q'')>0)$$

are always given by formulae of the form (1), (2), (3); only the right sides of the integral equations (2) are in every particular case to be modified according to the problem treated.

If $t_1 \leqq 0, \cdots, t_n \leqq 0$, that is, if at the instant of generation of call "0" all trunks are unoccupied, all terms on the right side of (1), save $V_0(0)$, disappear, so that then this formula takes

the simpler form $\Phi(q, \eta) = V_0(0)$.
While all reasoning on this memoir is conceived in such a way as is required for treating the many server problem $s > 1$, our formulae are valid also for $s = 1$.

The methods employed here apply also to all probability problems arising for a fully available bundle of trunks without waiting device, i. e., to the calculation of various loss-of-calls probabilities.

CHAPTER I. Construction of the Laplace-Stieltjes transforms of different distribution functions of waiting times.

In order to calculate the mathematical expectation $E e^{-q\tau_n}$ which, by the help of (4), yields the d.f. $\rho_n(t)$ we proceed as follows. Let X_n $(n = 0, 1, \cdots)$ be the instants of generation of our calls, so that $X_{n+1} - X_n = Y_n$ $(n = 0, 1, \cdots)$ and

$$X_n + t_{n1}, \quad \cdots, \quad X_n + t_{ns}$$

be the last s ends of conversations asked for before time X_n (and consequently having indices $< n$), taken in an arbitrary order. As there are s trunks, the waiting time τ_n of the n th call will be $\min(t_{n1}, \cdots, t_{ns})$, if this number is > 0, and zero in the opposite case. Writing

$$a^+ = \max(a, 0)$$

we have therefore

$$(5)\qquad \tau_m = \min_{\nu=1,2,\cdots,s}{}^{+}(t_{m\nu})$$

We now use the formula

$$(6)\quad \exp\left[-q \min{}^{+}(a_1,\cdots,a_s)\right] = 1 - \frac{1}{(2\pi i)^s}\int_{C_1}\cdots\int_{C_s} e^{\sum_1^s a_\nu z_\nu}\frac{q}{q+\sum_1^s z_\nu}\,\frac{dz_1\cdots dz_s}{z_1\cdots z_s}$$
$$(R(q)>0)$$

where the integration paths $C_1,\cdots,C_s$ are parallels to the imaginary axis, situated to the right of this axis, and traversed from below to above.

This formula shall be proven by induction. Supposing its validity for s - 1, we shall show that it is also valid for s ; as it is valid for s=1, it is true for all s.

For s = 1, (5) runs as follows:

$$e^{-qa^+} = 1 - \frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} e^{az}\frac{q}{q+z}\,\frac{dz}{z} \qquad (\delta>0, R(q)>0).$$

The integration path shall be shifted, while remaining parallel to itself, indefinitely towards the right for $a \leq 0$, and towards the left for $a>0$. Thus, we see that for $a\leq 0$ this integral is zero, but for $a>0$ it is equal to the sum of residues in $z=0$ and $z=-q$, that is to $1-e^{-qa}$, and this proves (5) for s = 1.

For $s>1$, we can admit that $a_s \geq a_1, a_2, \cdots, a_{s-1}$.

As multifold integrals of this type are absolutely convergent, which is shown in supplement I, we can treat (5) as an iterated integral. Proceeding in the same manner as previously, we find that

$$\frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} e^{a_s y_s}\,\frac{q}{q+\sum_1^{s} y_\nu}\,\frac{dy_s}{y_s} = \begin{cases} 0 & (a_s \leq 0) \\ \dfrac{q}{q+\sum_1^{s-1} y_\nu} - \dfrac{e^{-a_s\left(\sum_1^{s-1} y_\nu + q\right)}}{q+\sum_1^{s-1} y_\nu} & (a_s > 0)\,. \end{cases}$$

Thus for $a_s = 0$ formula (5) is evidently right. For the contribution of the second term to the right to the remaining (s - 1) fold integral is 0 because in $e^{\sum_1^{s-1}(a_\nu - a_s) y_\nu}$ all exponents $a_\nu - a_s$ are by hypothesis ≤ 0, while in virtue of the first term $\frac{q}{q+\sum_1^{s-1} y_\nu}$ the remaining $(s-1)$ fold integral has the same form, with $(s-1)$ instead of s, as the integral (5). Thus, because for $a_s > 0$, $\min^+(a_1, \cdots, a_{s-1}) = \min^+(a_1, \cdots, a_s)$, we have transformed (5) into the analogous formula for $s-1$ which is true by hypothesis. Therefore, (5) is proven for all $s \geq 1$.

In virtue of (5), we get

(6a)

$$e^{-q t_m} = 1 - \frac{1}{(2\pi i)^s}\int_{C_1}\cdots\int_{C_s} e^{\sum_1^s t_{m\nu} y_\nu}\,\frac{q}{q+\sum_1^s y_\nu}\,\frac{dy_1\cdots dy_s}{y_1\cdots y_s}$$

This formula shows in an evident manner that $e^{-\xi\tau_n}$ is a symmetric function of the $t_{n\nu}$.

We want to calculate $E\,e^{-\xi\tau_n} = E(e^{-\xi\tau_n} \mid t_{o\nu})$.

The waiting time τ_n is a definite function of the chance variables $T_0, \cdots, T_{n-1}$ (holding times) and $Y_0, \cdots, Y_{n-1}$ (interarrival intervals between successive calls) and of the given parameters $t_{01}, \cdots, t_{0s}$ which describe the initial conditions at the instant of generation X_0 of call "0". Therefore, replacing in all integral formulae T_ν and Y_ν by the corresponding small letters, we have

$$(7) \quad E(e^{-\xi\tau_n} \mid t_{o\nu}) = \int_0^\infty df_1(t_0) \int_0^\infty df_2(y_0) \cdots \int_0^\infty df_1(t_{n-1}) \int_0^\infty df_2(y_{n-1})\, e^{-\xi\tau_n}$$

where $e^{\xi\tau_n}$ must be expressed by the $t_{n\nu}$ according to (6a)

It is easily seen that the quantities $t_{n\nu}$ are equal, save for their order, to the quantities

$$(8) \quad \min_{\nu=1,\dots,s}{}^{+} t_{n-1,\nu} + T_{n-1} - Y_{n-1}\,, \quad \min_{\nu=1,2,\dots,s}^{(2)} t_{n-1,\nu} - Y_{n-1}\,, \cdots, \max_{\nu=1,\dots,s} t_{n-1,\nu} - Y_{n-1}$$

where $\min \leq \min^{(2)} \leq \min^{(3)} \leq \cdots \leq \min^{(s-1)} = \max_{\nu=1,\dots,s} t_{n-1,\nu}$.

As the numbers $t_{n-1,\nu}$, which are relative to the calls of indices $< n-1$, depend only on the chance variables $T_0, Y_0, \cdots, T_{n-2}, Y_{n-2}$, but not on T_{n-1} and Y_{n-1}, we have

$$(9) \qquad E(e^{-g\tau_m} | t_{m-1,\nu}) = \int_0^\infty df_1(t_{m-1}) \int_0^\infty df_2(y_{m-1})\, e^{-g\tau_m} ,$$

where $e^{-g\tau_m}$ is given by (6). We substitute for the $t_{m\nu}$ the expressions(8), which shows that (9) is a symmetric function of the $t_{m-1,\nu}$, and can then perform the integrations indicated in (9) under the integral signs by replacing factors of the form $e^{t_{m-1}\gamma}$ or $e^{y_{m-1}\gamma}$ respectively by

$$(10) \qquad \varepsilon_1(\gamma) = \int_0^\infty e^{t\gamma}\, df_1(t) \quad \text{and} \quad \varepsilon_2(\gamma) = \int_0^\infty e^{y\gamma}\, df_2(y)$$

After these operations we represent (9) as a sum of Fourier integrals in the real variables $t_{m-1,\nu}$, which is done by virtue of different transformations explained in supplement III. In the same manner we can calculate the conditional mathematical expectation

$$E(e^{-g\tau_m} | t_{m-2,\nu}) = \int_0^\infty df_1(t_{m-2}) \int_0^\infty df_2(y_{m-2})\, E(e^{-g\tau_m} | t_{m-1,\nu})$$

and represent it as a sum of Fourier integrals in the variables $t_{m-2,\nu}$.

Repeating this process $(n-2)$ times, we get finally for $E(e^{-s\tau_n} | t_{0\nu})$ which is the Stieltjes-Laplace transform, under the given initial conditions $t_{0\nu}$, of $g_n(t) = \mathrm{Prob}(\tau_n < t)$ an expression of the following form

$$
\begin{aligned}
E(e^{-s\tau_n} | t_{0\nu}) = {} & v_{n0}(0) + \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \Big(\sum_{\nu=1}^{n} e^{z_1 t_{0\nu}} \Big) v_{n1}(z_1; z_1) \frac{dz_1}{z_1} \\
& + \frac{1}{(2\pi i)^2} \int_{-i\infty+0}^{i\infty+0} \int_{-i\infty+0}^{i\infty+0} \Big(\sum\sum_{1 \le \nu_1 < \nu_2 \le n} e^{z_1 t_{0\nu_1} + z_2 t_{0\nu_2}} \Big) v_{n2}(z_1, z_2; z_1 + z_2) \frac{dz_1 dz_2}{z_1 z_2} \\
& + \cdots \\
& + \frac{1}{(2\pi i)^n} \int_{-i\infty+0}^{i\infty+0} \cdots \int_{-i\infty+0}^{i\infty+0} e^{\sum_1^n z_\nu t_{0\nu}} v_{nn}(z_1, \ldots, z_n; \sum_1^n z_\nu) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} ,
\end{aligned}
\tag{11}
$$

where $\int_{-i\infty+0}^{i\infty+0}$ signifies that the path of integration is parallel to, and situated at the right of, the imaginary axis of the complex plane.

The functions $v_{m\lambda}$ are defined by the initial conditions and the recurrence formulae

$$v_{0\lambda}(z_1,\dots,z_\lambda;y) = \delta_{\lambda 0}\,\frac{q}{q+y} \qquad (\lambda = 0,1,\dots,s-1),$$

$$
\begin{aligned}
v_{j+1,\lambda}(z_1,\dots,z_\lambda;y) &= \varepsilon_2(-y)\,v_{j\lambda}(z_1,\dots,z_\lambda;y) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\varepsilon_1(\zeta)-1}{\zeta}\,\cdot \\
&\quad \cdot\Big(y+\sum_1^\lambda z_\nu+\zeta\Big)^{-1}\Big[\Big(y-\sum_1^\lambda z_\nu\Big)\varepsilon_2(-y)\,v_{j,\lambda+1}(z_1,\dots,z_\lambda,\zeta;y) \\
&\quad - \zeta\,\varepsilon_2\Big(-\sum_1^\lambda z_\nu-\zeta\Big)\,v_{j,\lambda+1}\Big(z_1,\dots,z_\lambda,\zeta;\sum_1^\lambda z_\nu+\zeta\Big)\Big]\,d\zeta
\end{aligned} \tag{12}
$$

$$(\lambda = 0,1,\dots,s-1\ ;\quad j = 0,1,2,\dots),$$

$$v_{js}(z_1,\dots,z_s;y) = -\sum_{\lambda=0}^{s-1}\ \sum_{1',\dots,\lambda'=1}^{s} v_{j\lambda}(z_{1'},\dots,z_{\lambda'};y) \qquad (j = 0,1,2,\dots).$$

Posing $V_\lambda(z_1,\dots,z_\lambda;y) = \sum_0^\infty z^j\,v_{j\lambda}(z_1,\dots,z_\lambda;y) \quad (\lambda = 0,1,\dots,s)$, multiplying (11) by z^m, and the last two formulas respectively by z^{j+1} and z^j and summing, we obtain the formulae contained in the introduction. These formulae reduce the calculation of $\rho_m(t)$ essentially to the resolution of the system of equations (2), (3), and this problem is treated in chapters IV and V. In a manner analogous to (4), the composite probabilities

$$\rho_{m,m'}(t,t'\,|\,t_0) = \mathrm{Prob}\,(\tau_m < t,\ \tau_{m+m'} < t'\,|\,t_0)$$

can be expressed by means of the mathematical expectations

$$(13)\quad E(e^{-q\tau_n - q'\tau_{n+n'}} \mid t_{0\nu}) = E\left[e^{-q\tau_n} E(e^{-q'\tau_{n+n'}} \mid t_{n\nu}) \mid t_{0\nu}\right]$$

Replacing in (11) q, $t_{0\nu}$ and n respectively by q', $t_{n\nu}$ and n', we have

$$E(e^{-q'\tau_{n+n'}} \mid t_{n\nu}) = v_{n'0}(0) + \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\left(\sum_{\nu=1}^{s} e^{t_{n\nu} z_1}\right) v_{n'1}(z_1; z_1)\frac{dz_1}{z_1}$$

$$+ \cdots + \frac{1}{(2\pi i)^s}\int_{-i\infty+0}^{i\infty+0}\cdots\int_{-i\infty+0}^{i\infty+0} e^{\sum_1^s t_{n\nu} z_\nu}\, v_{n's}(z_1,\dots,z_s; \sum_1^s z_\nu)\frac{dz_1\cdots dz_s}{z_1\cdots z_s}.$$

In accordance with (13), we must now multiply this expression by $e^{-q\tau_n}$ (eq. (6a)) which is also a Fourier integral in the same variables $t_{n\nu}$ and represent this product as a sum of Fourier integrals in the $t_{n\nu}$. This can be done by means of formula (3.25) of suppl. III. Next we have to carry out the operation $E(\quad \mid t_{0\nu})$ (see (7)) which yields an expression of the form (11) with recurrence formulae of the form (12), the only difference being that the initially given $v_{0\lambda}$ are replaced by other expressions.

Likewise the generating function

$$\sum_{n=0,\, n'=0}^{\infty} z^n z'^{n'} E(e^{-q\tau_n - q'\tau_{n+n'}} \mid t_{0\nu})$$

is given by a sum of integrals of the form (1), with integrands V_λ satisfying (3), and a system of integral equations of the form (2), although with different right sides.

By the same method more general composite probabilities such as $Prob\left(\tau_n < t, \tau_{n+n'} < t', \tau_{n+n'+n''} < t'' \mid t_{0\nu}\right)$ can also be reduced to the resolution of a system of integral equations of the form (2), (3).

Chapter II. Construction of the generating functions of different probabilities.

Denoting by p_{na} the probability that, at the instant of generation of the $(n+a)$th call, exactly a calls (generated after time X_0) be waiting, the generating function $\sum_{n,a=0}^{\infty} x^a z^n p_{na}$ is represented by means of the function $\Phi(\xi, z)$ (eq. (1)) defined and calculated previously.

The generating function $\sum_{\lambda=0}^{s} \sum_{n=0}^{\infty} x^\lambda z^n p_n^\lambda$ of the probabilities p_n^λ that, at the instant of production of the n th call, exactly $\lambda = 0, 1, \cdots, s$ trunks be occupied is also calculated. This problem can be reduced to the resolution of a system of linear equations differing from (2) only by their right sides and with $\int_{-\infty}^{\infty} \cdots d\xi$ replaced by $\lim_{N \to \infty} \int_{-iN}^{iN} \cdots d\xi$.

CHAPTER III. Problems concerning the distribution of the Markovian Parameters. Theory of the Phenomenon of Temporary Blocking.

The quantities $t_{n\nu}$ defined in the beginning of Chapter I can be considered as "Markovian parameters" relative to the instant X_n, because they characterize exhaustively all that which the knowledge of events concerning calls generated before the instant X_n can tell us about phenomena which are posterior to X_n.

We have seen previously that $\min\limits_{\nu=1,\dots,s}^{+} t_{n\nu}$ is the waiting time τ_n of the n th call, and have explained our method how to obtain the distribution function $\rho_n(t)$ of τ_n. More generally we can study the distributions of the quantities $\min\limits_{\nu=1,2,\dots,s}^{(\mu)+} t_{n\nu}$ which are the symbolic "waiting times" elapsing between X_n and the instant when at least μ trunks of the bundle become free of conversations asked for before time X_n.

By means of our method we reduce the calculation of the generating functions of the mathematical expectations (m. e.)

$$E\left(e^{-q \min\limits_{\nu=1,\dots,s}^{(\mu)+} t_{n\nu}} \mid t_{0\nu}\right) \qquad (\mu = 1,\dots,s)$$

and

$$E\left(e^{-q_1 \min\limits_{\nu=1,\dots,s}^{+} t_{n\nu} - q_2 \min\limits_{\nu=1,\dots,s}^{(2)+} t_{n\nu}} \mid t_{0\nu}\right) \qquad (R(q_1)>0,\ R(q_2)>0)$$

to the problem of resolution of a system of integral equations analogous to (2), (3).

In order to obtain the joint distribution of all s quantities $\min_{\nu=1,\cdots,s} {}^{(\mu)+}t_{m\nu}$ $(\mu=1,\cdots,s)$, it is necessary to calculate the m. e.

$$E\left[\exp\left(-\sum_{\mu=1}^{s} q_\mu \min_{\nu=1,\cdots,s} {}^{(\mu)+}t_{m\nu}\right) \Big| t_{o\nu}\right] \qquad \left(R(q_1)>0, \cdots, R(q_s)>0\right).$$

We do this by means of a class of three-indices-operators $T_m^{\lambda\mu}$ which we have used already to treat other problems of this kind, and for the particularities of which we refer to a previous paper (Application d' operateurs intégro- combinatoires dans la théorie des intégrales multiples de Dirichlet", Ann. Inst. H. Poincaré, v. 11, 1949, p. 113-133). For the calculation of the generating function of the last mathematical expectation the formulae (1), (2), (3) remain still valid; however, the right sides of (2) must be replaced by certain functions of $q_1, \cdots q_s$. We treat then the particular case

$$f_1(t) = 1 - e^{-t}, \quad \varepsilon_1(\gamma) = \int_0^\infty e^{\gamma t} df_1(t) = \frac{1}{1-\gamma},$$

in which these functions have a particularly simple form.

In the second part of this chapter we generalize the assumptions which were made up to now, in order to be able to take into consideration the so called phenomenon of temporary blocking. This phenomenon due to technical reasons has the following effect. From the instant when the n th call is assigned to a non-occupied trunk, this one and all other non-occupied trunks are blocked during a certain time θ_m.

We consider the quantities θ_n as stochastic variables, stochastically independent of all other variables $T_\lambda, Y_\mu, \theta_\nu$ save T_n and assume that

$$\text{Prob}\,(T_n < t,\ \theta_n < \theta) = f(t, \theta),$$

$f(t,\theta)$ denoting a given distribution function of two variables which is independent of n. As in the preceding chapters we suppose that

$$E(T) = \int_0^\infty t\, d_t f(t,\infty) < \infty.$$

By τ_n we designate now the waiting time imposed on the n th call by the sole effect of previous calls (of indices $< n$). To τ_n must be added the blocking delay θ_n, so that the n th conversation begins at time $X_n + \tau_n + \theta_n$.

The calculation of the $E(e^{-\xi\tau_n} \mid t_{\cdot\cdot})$ is somewhat more complicated than under our previous assumptions. For the generating function of these mathematical expectations we obtain again an expression of the form (1), but the functions V_λ are now determined by a system of linear equations, in the left sides of which there appear simple as well as double definite integrals, while the right sides are equal to $\delta_{\lambda 0} \frac{\xi}{\xi + \gamma}$.

Under the particular assumptions $f(t,\theta)=(1-e^{-t})\,g(\theta)$ which implies that T_m and θ_m are stochastically independent, and $f_2(t)=1-e^{-\gamma t}$, we have employed the above mentioned system of integral equations to calculate the quantity $V_0(0)$, to which

$$\sum_{m=0}^{\infty} \gamma^m E(e^{-\gamma\tau_m}\,|\,t_{0\nu})$$

reduces itself for $t_{01}\leq 0,\ \cdots,\ t_{0\rho}\leq 0$, (in "Réduction de differ nts problèmes concernant la probabilité d' attente au téléphone, à la résolution de systèmes d' équations intégrales", Ann. Inst. H. Poincaré, ... 1949, p. 135 - 173 (p. 165).

CHAPTER IV.

Resolution of the integral equations (2), (3) for $f_1(t)=1-e^{-t}$, that is, for $\mathcal{E}_1(\zeta)=\frac{1}{1-\zeta}$.

When the Laplace -Stieltjes transform $\mathcal{E}_1(\gamma)=\int_0^{\infty} e^{\gamma t}\,df_1(t)$ is a rational function, the integral equations (2) can be transformed into functional equations, for any $f_2(t)$ and its tranform $\mathcal{E}_2(\gamma)$. Postponing the study of the case of an arbitrary rational $\mathcal{E}_1(\gamma)$, we have treated in this chapter the case of the simplest rational $\mathcal{E}_1(\zeta)$ i. e of $\mathcal{E}_1(\zeta)=\frac{1}{1-\zeta}$ in a more thorough manner. Notice here, that putting $f_1(t)=1-e^{-t}$, we have $E(T)=\int_0^{\infty} t\,df_1(t)=\int_0^{\infty} t\,e^{-t}\,dt=1$ which signifies that we have taken the mean holding time as unity of times.

Now put $\varepsilon_1(\zeta) = \frac{1}{1-\zeta}$ in the integral equations (2). By virtue of the properties of the analytic functions $V_\lambda(\gamma_1, \cdots, \gamma_\lambda; y)$ explained in supplement IV the integrands appearing in (2) have then in the right half plane of the complex variable ζ a unique pole $\zeta = 1$ and are $O(|\zeta|^{-2})$ for $R(\zeta) > 0$, $|\zeta| \to \infty$. Accordingly the integrals in (2) are equal to the respective residues of their integrands at $\zeta = 1$, so that these equations are transformed into

$$\begin{aligned}(14)\quad & \left[1 - \gamma\varepsilon_2(-y)\right] V_\lambda(\gamma_1, \cdots \gamma_\lambda; y) - \gamma\Big(y - \sum_1^\lambda \gamma_\nu - 1\Big)^{-1} \Big[\Big(y - \sum_1^\lambda \gamma_\nu\Big)\varepsilon_2(-y) V_{\lambda+1}(1, \gamma_1, \cdots \gamma_\lambda; \lambda) \\ & - \varepsilon_2\Big(-\sum_1^\lambda \gamma_\nu - 1\Big) V_{\lambda+1}\Big(1, \gamma_1, \cdots, \gamma_\lambda; \sum_1^\lambda \gamma_\nu + 1\Big)\Big] = \delta_{\lambda 0} \frac{g}{g+y} \\ & (\lambda = 0, 1, \cdots, s-1).\end{aligned}$$

As has already been mentioned, in the simplest case studied in queueing theory, that is, when at the instant X_0 of generation of the initial call "0" all s trunks are non-occupied so that $t_{01} \leq 0, \cdots, t_{0s} \leq 0$, equation (1) becomes

$$\sum_{n=0}^{\infty} \gamma^n E(e^{-g z_n} \mid 0, \cdots, 0) = V_0(0);$$

therefore, we are principally interested in the construction of $V_0(y)$ and $V_0(0)$.

To that end we have to put in the equation (14) $\eta_1 = \cdots = \eta_s = 1$ (that is, to take all η_ν equal to the unique pole of $\varepsilon_1(\zeta)$). For the s + 1 functions

$$V_\lambda(y) = V_\lambda(1, \cdots, 1; y) \qquad (\lambda = 0, 1, \cdots, s) \tag{15}$$

we obtain in this manner from (14) and (3) the s + 1 linear non homogeneous equations

$$\begin{aligned} &[1-\eta\varepsilon_2(-y)]\,V_\lambda(y) - \frac{\eta}{y-\lambda-1}\big[(y-\lambda)\,\varepsilon_2(-y)\,V_{\lambda+1}(y) - \\ &\quad - \varepsilon_2(-\lambda-1)\,V_{\lambda+1}(\lambda+1)\big] = \delta_{\lambda 0}\,\frac{q}{q+y} \qquad (\lambda = 0, 1, \cdots, s-1), \\ &\sum_{\lambda=0}^{s} \binom{s}{\lambda} V_\lambda(y) = 0 . \end{aligned} \tag{16}$$

Therefore, $V_0(y)$ can be expressed as the quotient of two functions of y, the numerator depending linearly on the s still unknown quantities $V_\lambda(\lambda)$ and the denominator containing the factor

$$s - y - s\eta\,\varepsilon_2(-y) . \tag{17}$$

Posing $y = \lambda$ in (16) we get for the $V_\lambda(\lambda)$ the s - 1 homogeneous linear equations

$$[1-\eta\varepsilon_2(-\lambda)]\,V_\lambda(\lambda) - \eta\,\varepsilon_2(-\lambda-1)\,V_{\lambda+1}(\lambda+1) = 0 \qquad (\lambda = 1, \cdots, s-1). \tag{18}$$

It is easily seen, as a consequence of Rouché's theorem, that for $|\eta| < 1$ the function (17) has in the right half plane of the variable y a unique zero $y = y_0(z)$. Since all functions $V_\lambda(z_1, \cdots, z_\lambda; y)$, and hence the functions (15), are finite for $R(z_1) \geq 0, \cdots, R(z_\lambda) \geq 0, R(y) \geq 0$, the aforementioned numerator of $V_0(y)$ must also disappear for $y = y_0(z)$ and this property yields an s th linear relation which together with equations (18) permits us to calculate all s quantities $V_\lambda(\lambda)$. For $V_0(0)$ we obtain thus the formula

$$\text{(19)} \quad \sum_{m=0}^{\infty} z^m E(e^{-q\tau_m} | 0) = V_0(0) = \frac{1}{1-z}\left[1 + \frac{q}{q + y_0(z)} A_0^{-1}(z)\right] \quad (|z|<1),$$

where $A_0(z)$ is given by the equation

$$\text{(20)} \quad \begin{gathered} A_0(z) = y_0 \sum_{\lambda=0}^{s-1} \binom{s-1}{\lambda} \left[(y_0-\lambda)(y_0-\lambda-1) \prod_{k=1}^{\lambda} h(k)\right]^{-1} \\ \left[h(k) = \frac{z\, \varepsilon_2(-k)}{1 - z\, \varepsilon_2(-k)}, \quad y_0 = y_0(z)\right]. \end{gathered}$$

From (19) we obtain for the generating function $F(z)$ of the distribution functions $\rho_m(t) = \rho_m(t \mid 0)$ by virtue of equation (4)

$$(21)\quad F(z) = \sum_{m=0}^{\infty} z^m \rho_m(t) = \frac{1}{1-z}\left[1 + A_0^{-1}(z)\, e^{-y_0(z)t}\right]$$

$$(|z|<1,\ t>0)$$

One can obtain thus the distribution functions $\rho_m(t)$ by developing the right side in a Taylor series, and in accordance with the present assumption $t_{01} \leq 0, \cdots, t_{0s} \leq 0$ one sees that in particular $\rho_m(t) \equiv 1 \quad (m = 0, 1, \cdots, s-1)$.

In order to find the limit distribution function

$$(22)\quad \rho(t) = \lim_{m\to\infty} \rho_m(t)$$

we represent $\rho_m(t)$ as a complex integral, i. e.,

$$\rho_m(t) = \frac{1}{2\pi i}\int_K F(z)\frac{dz}{z^{m+1}}$$

where K denotes a little circle the center of which is $z = 0$.

If $f_2(t)$ is such that

$$\varepsilon_2'(0) = \int_0^{\infty} t\, df_2(t) > \frac{1}{s}\int_0^{\infty} t\, df_1(t) = 1,$$

there exists a constant $z^* > 1$ such that for $|z| < z^*$ the zero $y_0(z)$ of the expression (17) is a holomorphic function; the same is therefore true for $A_0(z)$ (equ. (20) and (21)).

Therefore, if $A_0(z)$ has no zero on the circumference $|z|=1$, which is true <u>if</u> the limit (22) exists, we find, extending K beyond the unit circle, that

$$\rho(t) = 1 + A_0^{-1}(1)\, e^{-\gamma_0(1)t}$$

and can moreover establish an asymptotic evaluation for the difference $\rho(t) - \rho_m(t)$. Now the existence (with a single exception) of the limit (22) has been proven, for arbitrary $f_1(t)$ and $f_2(t)$ by J. Kiefer and J. Wolfowitz. But here we need not resort to this theorem, for we prove that in the present case

$$\rho_m(t|0) \geq \rho_{m+1}(t|0) \qquad (m=0,1,2,\cdots),$$

so that the existence of the limit (22) is evident.

Also for $\int_0^\infty t\,df_2(t) \leq \frac{1}{s}\int_0^\infty t\,df_1(t)$, whence $\rho(t) \equiv 0$, it is possible to establish asymptotic formulae for $\rho_m(t)$.

The end of this chapter contains the construction, for $\mathcal{E}_1(\zeta) = \frac{1}{1-\zeta}$, of the function $V_1(z_1; z_1)$ which appears in (1), as well as several applications of the last formulae to the problems of Chapter II.

CHAPTER V. Resolution of the integral equations (2), (3) under different assumptions concerning the Laplace-Stieltjes transforms $\varepsilon_1(\zeta)$ and $\varepsilon_2(\zeta)$.

The first of the two cases, mentioned in the introduction, in which the equations (2), (3) can be resolved by means of a finite number of steps, is outlined under the assumption

$$f_1(t) = \sum_{i=1}^{m} c_i \left(1 - e^{-a_i t}\right) \qquad \sum_{i=1}^{m} c_i = 1, \quad a_1 \neq a_2 \neq \cdots \neq a_m$$

that is for

$$\varepsilon_1(\zeta) = \sum_{1}^{m} \frac{a_i c_i}{a_i - \zeta} \tag{23}$$

Proceeding in the same manner as on the occasion of the deduction of equation (14), we can replace then all integrals in (2) by the sums of the respective residues of their integrands in $\zeta = a_1, \cdots, a_m$. Thus these integral equations are transformed into functional equations, the method of solution of which is explained in detail for s = 2 and outlined for arbitrary s . In order to construct $V_0(y)$ for s = 2 , it is necessary to calculate a certain determinant $D_m(y,z)$ (given here for n = 2 and n = 3) and to determine those $\binom{m+1}{2}$ roots $y_\nu(z)$ $\left(\nu = 1, 2, \cdots, \frac{m^2+m}{2}\right)$ of the equation

$$D_m(y,z) = 0,$$

which for sufficiently small $|\eta|$ are situated in the right y-half-plane.

When the numbers $y_\nu(1)$ are all different, the limit distribution function $\lim_{n\to\infty} \rho_n(t) = \rho(t)$ is of the form

$$\rho(t) = 1 - \sum_{\nu=1}^{\frac{n^2+n}{2}} \alpha_\nu e^{-y_\nu(1)t} \qquad \left(t>0,\ s=2,\ \varepsilon_2'(0) > \tfrac{1}{2}\varepsilon_1'(0) = \tfrac{1}{2}\sum_1^n \frac{c_\nu}{a_\nu}\right).$$

For every s , the $V_\lambda(\eta_1,\cdots,\eta_\lambda; y)$ appearing in (2), (3) are, under the present assumptions, rational functions of y and $\varepsilon_2(-y)$.

The second case in which the equations (2), (3) can be resolved by means of a finite number of operations, is studied under the assumptions

(24)
$$\varepsilon_1(\zeta) = \sum_{\nu=0}^{n} a_\nu e^{\nu k \zeta} \qquad \left(\sum_0^n a_\nu = 1,\ a_n > 0,\ n \geq 1\right)$$
$$\varepsilon_2(\zeta) = \sum_{\mu=1}^{m} \frac{b_\mu c_\mu}{b_\mu - \zeta} \qquad \left(\sum_1^m c_\mu = 1,\ c_1 \neq 0, \cdots, c_m \neq 0,\ R(b_1) > 0, \cdots, R(b_m) > 0,\ m \geq 1\right)$$

The method of resolution of (2), (3) is explained in detail for s = 2 and then outlined for arbitrary s ; in every case the V_λ are rational functions of y. In order to construct $V_0(y)$ for s = 2 , it is necessary to calculate those $n(n+1)$ zeros $y_\nu(\eta)$ of another determinant function $D_n(y,\eta)$ (given here for n = 1, 2, 3) which for sufficiently small $|\mathfrak{z}|$ are situated in the left y - halfplane.

Finally we suppose that from the initial instant X_0 an enormous number of calls is generated. Then, of course, $\lim_{m\to\infty} \rho_m(t) = 0 \quad (t \geq 0)$, but we nevertheless study this case in order to be able to establish for $\rho_m(t)$ asymptotic expressions for $m \to \infty$.

Our suppositions can best be rendered by assuming that

(25) $f_2(t) = 0 \quad (t < 0), \qquad = 1 \quad (t > 0),$ so that $\varepsilon_2(\zeta) = 1.$

Then the form of the integral equations (2) as well as the theory of their resolution under both hypotheses for $\varepsilon_1(\mathfrak{z})$ treated in this chapter is considerably simplified. We find that, for an $\varepsilon_1(\zeta)$ according to (23) as well as under the hypotheses (24), our integral equations can be resolved by resolving several systems of linear algebraic equations, without having to resort to certain solutions of transcendental equations.

In order not to interrupt too often the exposition of our theory, several demonstrations have been placed at the end of this memoir (supplements S1 - S7).